THE KEPLER PROBLEM

Due date: Jan 31, 2024 — 9am

A two-body problem in a central potential can be described analytically. The orbital equation cannot be written explicitly as a function of time (dr/dt), but it can be written explicitly as a function of angle (dr/dv). In practice, however, we frequently need to compute the location of a moving object as a function of time. In such cases we resort to two possible approaches: solve the problem iteratively, with anomalies, or solve the differential equation numerically. We will apply both methods to the motion of Halley's comet and to study the long-term (numerical) stability of its orbit.

1. **Iterative solution.** Halley's comet might be the most famous comet of all time. Its perihelion distance is 0.587 au, eccentricity is 0.967 and the orbital period is 76 years. Using the Newton-Raphson method, compute the position of Halley's comet around the Sun as a function of time. Plot the orbit.

2. **Runge-Kuta approximation.** Compute the coefficients α_1, α_2, β_1, β_2, β_3, γ_1, γ_2 and γ_3 for the third order of Runge-Kutta expansion. Typeset the derivation, mostly for LaTeX practice.

3. **Differential solution.** Using an off-the-shelf 4$^{\text{th}}$ order Runge-Kutta method, compute the orbit of Halley's comet and compare it with the orbit obtained by the iterative solution. Plot the accumulated error as a function of time (or the number of orbits).

4. **Stability.** Study the stability of Halley's comet using several differential integrators. Be sure to explicitly reference the used code or algorithms.

A quick recap: the equation of an elliptical orbit is:

$$r(v) = \frac{a(1-e^2)}{1+e\cos v}, \tag{1}$$

where a is the semi-major axis, e is orbital eccentricity and v is true anomaly. The true anomaly is measured from perihelion and can be related to time via the eccentric and mean anomalies:

$$\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}, \qquad M = E - e\sin E = \frac{2\pi(t-\tau)}{P}, \tag{2}$$

where E is the eccentric anomaly, M is the mean anomaly, τ is the reference time at perihelion and P is the orbital period. Equation $M = E - e\sin E$ cannot be solved analytically for E, thus it needs to be solved iteratively, using the Newton-Raphson method. This is also the reason why $r(t)$ cannot be written out explicitly.