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CURVFAM: The First Steps to a Computational Universal Curvature Fitting Algorithm

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ABSTRACT

With the current power of modern computers, computational modeling and fitting has become the de facto choice in the analysis of astrophysical data. Here I present the logical framework and baseline code for CURVFAM (CURVature Fitting AlgorithM), a tool that models and analyzes the effects of universal curvature. Unlike current universe simulators, this code is optimized for solely curvature modeling in an effort to conserve time and resources. Future implementations of this code will allow for the determination of universal curvature based on real world observations. This paper gives an overview of the scientific reasoning and formulations behind the program as well as a look into what the future of CURVFAM will offer.

1. INTRODUCTION

The measure of an object's flatness is solely a question of the object's geometry. An object can have one of three types of curvature that can have (in cases of extreme curvature) significant effects on measurements. The three types of curvature are: flat (zero curvature), open (negative curvature), and closed (positive curvature). On terrestrial scales, an object's curvature can easily be measured by observing the effects of two initially parallel lines and the sum of the interior angles of a triangle. On a flat (euclidean) surface (e.g. a table top), parallel lines will forever run equidistant from each other and never intersect while the interior angles of a triangle will add up to 180°. On a closed surface (e.g. the surface of the Earth), lines that start parallel will eventually intersect and the interior angles of a triangle will add up to greater than 180°. On an open surface (e.g. a saddle), parallel lines will eventually diverge and the interior angles of a triangle will add up to less than 180°.

The question arises of the shape of our own reality. The quest for the global shape of the universe has major implications in both history and the future of astrophysics. For example, it is widely believed that a closed universe will ultimately end up recollapsing (Zel'dovich & Grishchuk 1984; Barrow & Tipler 1985). In a more historical realm, properties of the primordial universe, before inflation, can be inferred from current curvature parameters(Aslanyan & Easther 2015). This also gives us a more detailed view of what happened during inflation and maybe what caused it.

In an attempt to determine the shape of the universe and how it might have evolved over time, some groups of researchers have taken to observing the equations that govern the shape of the universe (the Friedmann–Lemaître-Robertson–Walker (FLRW) metric) and analyzing the possible universal results (Vardanyan et al. 2011; Yu & Wang 2016; Mortsell & Jonsson 2011). Attempts range from directly solving the equations (Odintsov & Oikonomou 2019; Steiner 2007) to analyzing many different models using statistics (Vardanyan et al. 2011). While the shape of the universe (and in particular the FLRW metric) involve parameters like the density of matter Ω_M and dark energy Ω_Λ, there is one value that is derived from the density parameters and can therefore be used to determine those fundamental values. This value is the curvature parameter K.

The two main methods used for the determination of the curvature parameter utilize supernovae and the cosmic microwave background (CMB). The first method involves comparing data from supernova surveys with solutions to the FLRW metric across various tested values for Ω_M and Ω_Λ (Inserra et al. 2021; Wang et al. 2005). Alternatively, other studies employ power spectra of the irregularities in the CMB to statistically estimate curvature (Vardanyan et al. 2011). While each method of determining the density parameters yields plausible results, there is still tension between the various methods on the curvature value and even the resulting shape of the universe (Handley 2021).

The method of prescribing different models to the universe and statistically analyzing which fits observed qualities of the universe better is typically used for constraining values such as the Hubble constant (H_0), the current matter density parameter (Ω_M), and the current dark energy density parameter (Ω_Λ) to name a few (Inserra et al. 2021; Gupta 2019). It can, however, be extrapolated further to analyze additional parameters or even the shape of the universe as a whole.

A new approach to determining the global structure of the universe has been gaining popularity with the ever increasing processing power of modern computers. This approach models an entire universe of particles and subjects them to the laws of physics allowing for the system to evolve with time. At first, such simulations (called N-body simulations) were used to observe galaxy clustering (Aarseth et al. 1979), but more recently, these simulations are modeling universes with trillions of particles to high degrees of accuracy (Maksimova et al. 2021). The implications of such detailed simulations is the ability to directly measure the desired parameters ($\Omega_M,\ \Omega_\Lambda,\ H_0$) from the simulated universe.

The difficulty with the universal simulations is the amount of computational power and time required to process such large amounts of calculations. This makes the use of such simulations excessive for solely determining curvature parameters. In the case of curvature modeling, a simulation more specifically designed with curvature in mind might provide a more appealing option.

In this paper I present a new method for determining the global curvature of the universe. I introduce the logical framework and baseline code for a curvature fitting algorithm (CURVFAM). I summarize the theory behind this method in Section 2. In Section 3 I explain a lower dimension model as a simpler introduction to the more general case. I follow this with a more comprehensive model in Section 4. Future addendums to the code as well as streamlined processes are discussed in Section 5 followed by a concluding summary in Section 6.

2. THEORY

There are two overarching methods to determining the properties of the universe: a bottom-up approach and a top-down approach. In a bottom-up approach, observables and parameters are measured from objects existing in the universe (e.g. supernovae and quasars). An observable, as used here, is defined as anything that we measure about an object (distance, size, temperature, color, angle, etc.). A theory is then applied to the observables to determine the properties of the universe. In a top-down approach, the properties of the universe are assumed to be known and a value or function of values are applied to them to determine what values the observables would take on. In direct contrast to a bottom-up approach, a top-down approach is calculating observables with the large-scale properties of the universe already known (or assumed).

Here, I apply a top-down approach to determine the effects of any arbitrary curvature function for the observable universe. I start by assuming that the curvature metric K can change as a function of both time, $T(t)$, and position in space, $\Upsilon(\vec{r})$ ($\vec{r}$ is the position vector of the object). In Equation 1, $\Upsilon(\vec{r})$ represents the spatial dependency of curvature (ie. if curvature varies based on location in the universe), and $T(t)$ represents the time dependency (ie. the allowance for curvature to change over time). Since we cannot assume that we know a set of universal coordinates (the classic Copernican principle) (Clarkson et al. 2008), the spatial dependence will be used here in terms of coordinates relative to an observer (realistically spherical coordinates).

$$K(\vec{r}, t) = \Upsilon(\vec{r})T(t) \tag{1}$$

Typically, the curvature K would be determined by solving the FLRW metric. However, should the FLRW metric prove too complex to analyze or if the data does not provide adequate information and parameters for the metric, a series of exponentials would provide a useful approximation. Exponentials make a valid approximation because of their unbounded nature. When used with variable coefficients, exponentials can take on any value and, when used in a series, can model any function.

By prescribing a known curvature metric to a universe of test data, observables can be measured with respect to a chosen reference point. In this case, the observables will be: perceived distance to any object from the reference point and the measured angle formed between two observed objects with respect to the reference point.

The important distinction to make for observations in curved space is that the distance between any two objects is going to change depending on the curvature. As such, the apparent angle between the two objects (from the reference point) is going to increase or decrease based on the curvature. Figure 1 depicts an example of how two objects in flat space will be perceived at different locations in a space of closed curvature. As the space between the objects is stretched, the perceived angle (φ) between the objects is increased. Along with this change in perceived angle the distance to the objects may change as well as the space between the objects and the reference point is stretched.

By determining how the observables change for a set of objects with known, fixed locations, the dependence of observables on the curvature is shown. With the top-down model constructed, the curvature function (Equation 1) can be optimized for a curvature that yields observables that we see in our own universe.

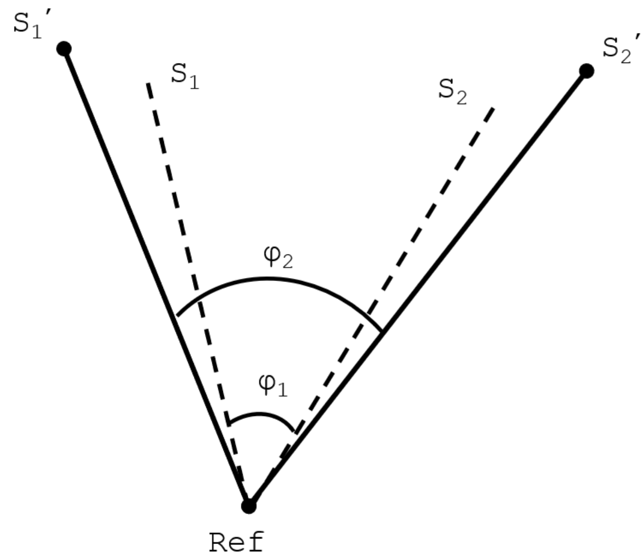

Figure 1. An example of how the perceived observables might change in a closed space. The locations of two objects in flat space (S_1 and S_2) are perceived at different locations (S_1' and S_2') when viewed in a closed space.

3. 1-D MODEL

Before designing a model that incorporates all dimensions, I first apply the math to a simpler one-dimensional model to see how curvature affects the observable on geometries that are easier to both comprehend and simulate.

For a one-dimensional model, the only observable would be the distance from the reference point along the direction of space (r). Here I am considering the reference point to be the zero point of the space. This is a fair assumption to make since any observer would see themselves at the origin of their own reference frame when making observations. To model this, I generate a set of uniformly distributed points in flat space (points evenly spaced along a straight line), then generate their positions in curved space by applying a prescribed curvature function to their locations. In euclidean space, this model would be a straight line. The curvature metric in this model would be only a function of position. Figure 2 shows how the positions of the objects in flat space are transcribed as new positions in an example curved space.

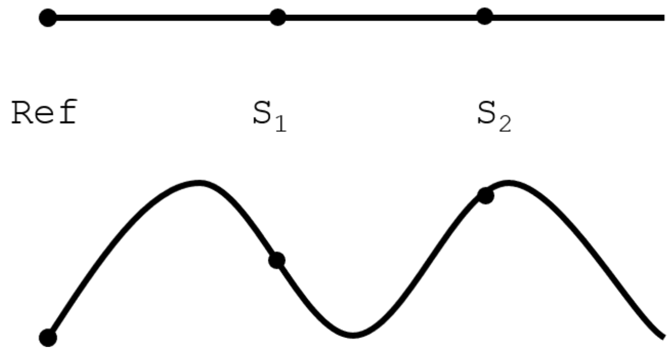

Figure 2. Sinusoidal example of how one-dimensional distance changes as the curvature of space changes. In flat space the distance traveled is a straight line, but in curved space the distance traveled is a path length along the curvature function.

For any one-dimensional space, the distance between any two points is given by the distance formula for curves. Equation 2 shows the distance r to any object S relative to the reference point along any path (straight or curved). Here $|K'|$ represents the modulus of the first derivative of the curvature function.

$$r = \int_0^S \sqrt{1 + |K'|^2}\, dr \tag{2}$$

To give a more explicit example, I prescribe the curvature polynomial given in Equation 3.

$$K = 2r^2 - 4r \tag{3}$$

It is important to recall that for the purpose of this example case, a polynomial is a less than realistic interpretation of the universe. I use it here to emphasize the effects of the curvature. Figure 3 displays how the distribution of distances changed from uniform spacing and density to a less uniform distribution when acted on by curvature.

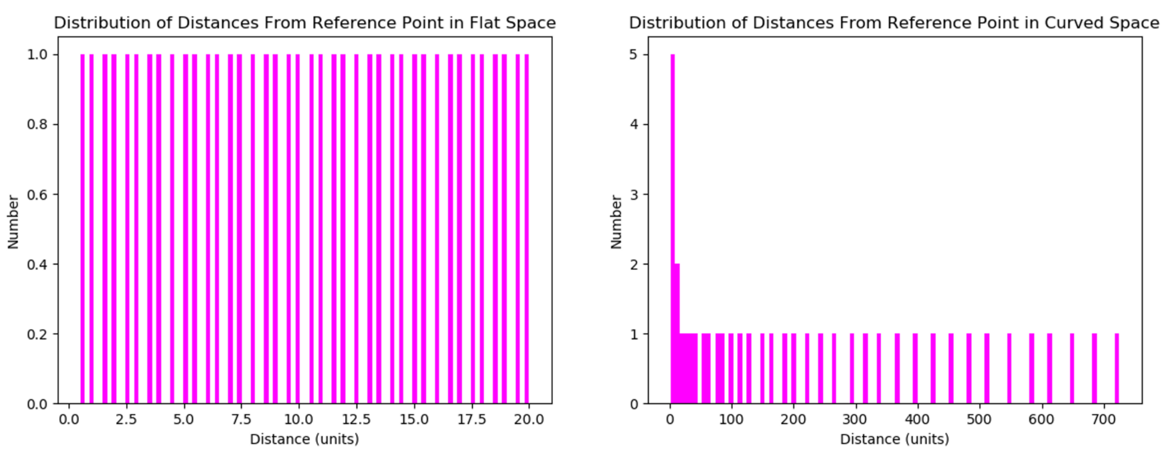

Figure 3. How the distribution of one-dimensional distances from the reference point (zero) changes when curvature is applied. It is important to note that the exact results depend on the curvature functions used.

4. COMPREHENSIVE MODEL

In transitioning to higher spatial dimensions, it is important to note that the reference point is no longer at the zero point of a function of curvature. Rather, the reference point is now at the origin of a three-dimensional universe that is uniformly populated with points out to a finite location. The universe starts flat with the objects distributed homogeneously and isotropically throughout the volume. I choose to start with this because on the universal scale, the distribution of galaxies is found to be homogeneous and isotropic (Barrow & Matzner 1977). When applying supernova data to the model, the imported data is unlikely to be either, however they are representatives from a universe that is both. A curvature metric is prescribed to the space. Unlike the one-dimensional case, the curvature metric here is a series of three equations, each describing the effects of curvature on a single observable. Spherical coordinates (φ, θ, r) are chosen due to the fact that all observations are relative to a single reference point which I consider the origin here. Equation 4 shows how the curvature metric is broken into three separate functions to represent to the three observables. Υ represents the positional part of the observables (φ, θ, and r) while the time component T is covered by the r observable. The r observable represents the distance along the line of sight from the object to the observer. However, because light has a finite speed, this distance becomes linked with time. Therefore, the further away the object, the further back in time we observe it and the curvature at the time of the light's emission must be considered.

$$\begin{aligned} K_\varphi(\varphi,\theta,r) &= \Upsilon(\varphi,\theta,r)T(r) \\ K_\theta(\varphi,\theta,r) &= \Upsilon(\varphi,\theta,r)T(r) \\ K_r(\varphi,\theta,r) &= \Upsilon(\varphi,\theta,r)T(r) \end{aligned} \tag{4}$$

For an easily perceived example, I apply a closed curvature metric that is a series of simple polynomials. The curvature metric I use is given in Equation 5. Again, a polynomial is not the most realistic approach to modeling the actual universe, but for the purposes of demonstration it provides an easy to view example.

$$\begin{aligned} K_\varphi &= \varphi^2 + r\theta \\ K_\theta &= \varphi^2 - \varphi\theta \\ K_r &= \varphi^2 - \varphi - \theta \end{aligned} \tag{5}$$

All the observables (φ, θ, and r) are possible functions of each other. It is in this sense that CURVFAM accounts for local curvature as well as grand scale curvature. The prescribed curvature metric is applied to each object's position vector resulting in a new primed position vector that gives the object's location in curved space with regard to the euclidean space. For example, an object with observables $(\varphi, \theta, r) = (1, 2, 5)$ would become $(11, -1, -2)$. It is important to note that the curvature functions are calculated using φ and θ in radians so any value initially in degrees is first converted to radians. The analysis between flat and curved can be made as computationally simple as possible with all calculations being similar with the difference being the data set used (primed vs unprimed). Because the program calculates the primed locations and then determines the observables, local

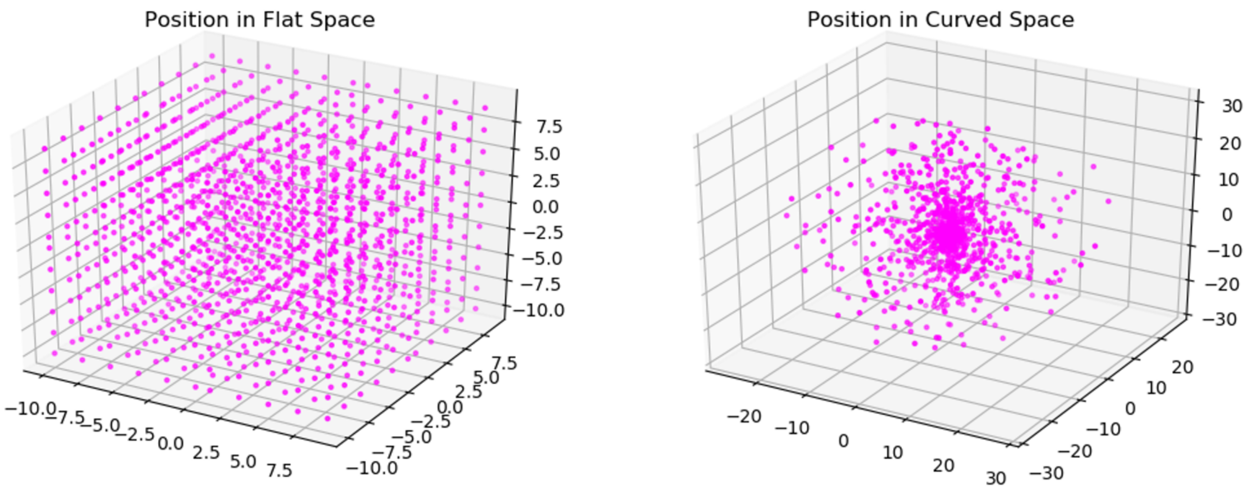

Figure 4. How the three-dimensional uniform distribution is affected by the spatial curvature example in Equation 5. The reference point is at the center of the distribution (the origin). It is important to note that the exact results depend on the curvature functions used.

curvature as well as any curvature between the objects and the reference point are accounted for. Figure 4 shows how curvature causes the observed spatial distribution of points to warp.

The change in spatial distribution also changes the observed angle between any two points. To determine the effects of curvature on the observed angles, I considered every pair of two adjacent data points. Adjacent is defined here as two objects that, in flat space, lie directly next to each other along a single Cartesian dimension (x, y, z) and differ by one separation distance in only one coordinate (eg. (1,1,1) is adjacent to (1,2,1) but is adjacent to neither (1,2,2) nor (1,3,1)). The angle between two objects is calculated using the dot product of their respective position vectors. I use the angle distribution and the average angle value for flat space as a reference against which to compare the distribution and average angle value for the same pairs in curved space.

Recall that the primed positions of the objects is being projected into euclidean space. Therefore, the angle between a pair of objects in curved space is calculated the exact same way as in flat space (with the primed position vectors this time). The curvature metric is being used to calculate primed locations of the objects and is already taking curvature into account and transcribing the curved space into flat space.

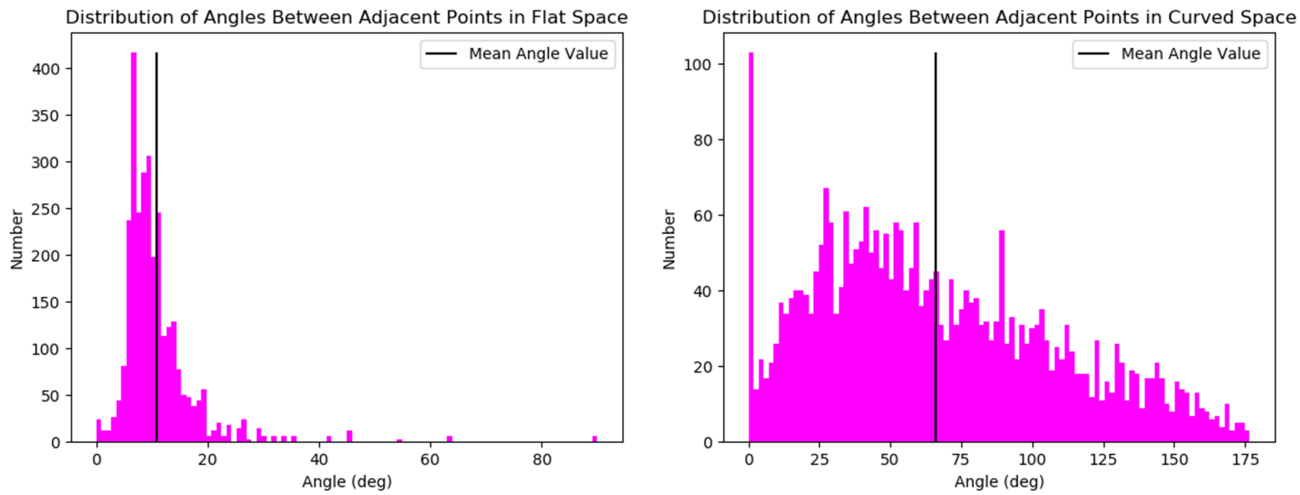

Figure 5. How the distribution of angles as well as the average angle value is affected by the spatial curvature example in Equation 5. It is important to note that the exact results depend on the curvature functions used.

Figure 5 shows how the same curvature that was applied to produce Figure 4 affects the observed angles between previously adjacent pairs. In flat space, the angles between pairs is a very narrow distribution. This is expected due to the uniform nature of the points. When curvature is applied, the distribution loses the initial spatial uniformity and therefore the initial angular uniformity causing the average angle between previously adjacent points to change. Generally this will be observed in a spreading of the distribution and an increase in the mean angle for closed space and a decrease in the mean angle for open space. For the test case here, the distribution spreads out and the mean angle increases from 10.92° to 66.14°.

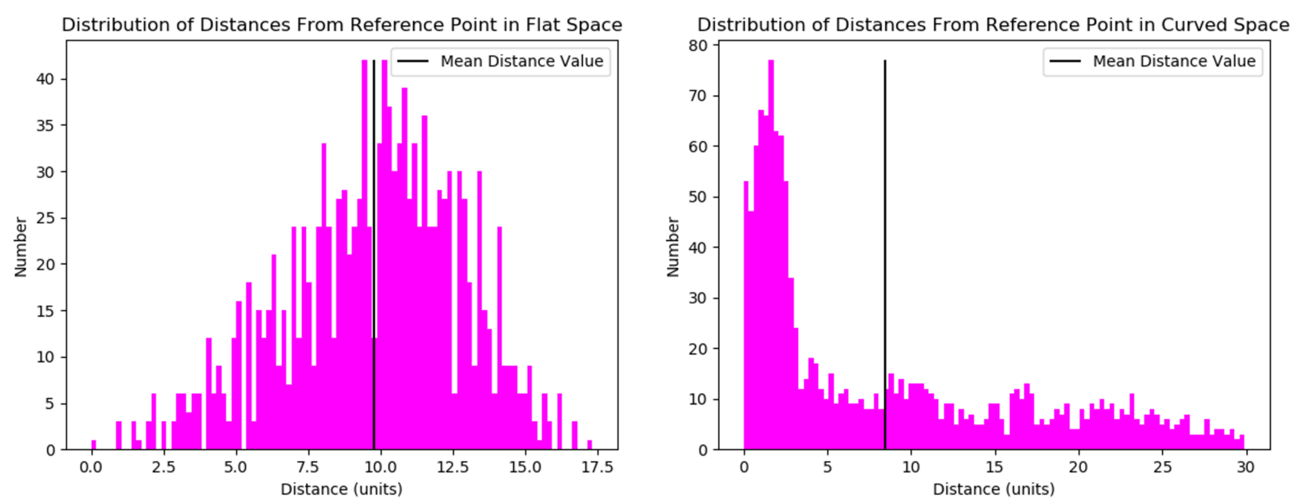

Figure 6. How the distribution of distances as well as the average distance value is affected by the spatial curvature example in Equation 5. It is important to note that the exact results depend on the curvature functions used.

The last observable to examine is the radial distance to the objects. The curvature parameter is also a function of time which is described by this distance. Figure 6 displays how curvature can affect the apparent distance between an object and the reference point. The distance in flat space is a near symmetrical distribution representative of the initial cubic uniformity. When curvature is applied, the distances to the points no longer appears as symmetrical as the initial uniformity is lost. In the case presented here, the average distance to an object in flat space is 9.79 units while in the curved space it becomes 8.44 units. While this is not as drastic of a change as the angular distribution, it does still exemplify how the perceived distances can change as a result of curvature.

As I have demonstrated here with this test case, CURVFAM is able to calculate the angular distributions as well as distance distributions for selected data (in this case uniform population) for a prescribed curvature metric. In its current state, this tool is helpful for observing the effects of various curvatures on observed quantities. This provides a teaching tool to display the effects of different curvature styles (open/closed) as well as the effects of various curvature strengths. I now turn to addressing the future fitting aspect of the program.

5. FUTURE IMPLEMENTATIONS

At it's current stage, CURVFAM does not yet posses a finalized fitting component. Future versions of CURVFAM will include a fitter that allows for the interpretation of real-world observational data. The fitter will utilize the same top-down method of describing curvature in order to determine which curvature metric best fits imported observable data. The difference is that rather than applying curvature to a set of uniform points, the observational data will represent the data points in curved space. The fitter is then tasked with finding the best curvature functions to produce the observed results. An additional component that will be added to future versions is the inclusion of general relativity. When considering universal curvature, general relativity plays a significant part in the determination of universal parameters such as density parameters and values such as the Hubble constant. As such, the fitter will take relativity into account and return a curvature fit as well as possible universal parameter values.

With the large amounts of supernova data available for use publicly (Guillochon et al. 2017) as well as the possibility of future supernovae surveys, the ability to process large amounts of data with speed and accuracy is important. CURVFAM already demonstrates the ability to handle thousands of data points in this current top-down state. However, with the addition of the fitting algorithm, a decrease in efficiency is to be expected. As such, new methods of modeling as well as new fitting algorithms will be continually tested and applied to CURVFAM.

In its current state, CURVFAM works as a client side code that must be directly modified to include the local data files. In future implementations we will apply a more user friendly front end GUI to help streamline data input and output. This will allow for the code to be accessed as either a package or program without the need to directly edit the source code. Stemming off from this, even further versions of CURVFAM can be modified to be presented in a web-based format. This will allow for the intense calculations to be run on more efficient servers and provide additional ease-of-use for the user.

6. CONCLUSION

In this paper I presented the current state of a continuing process towards unifying computational modeling and universal curvature fitting. CURVFAM is a code that allows for the determination of observables in a prescribed curved space. The implication of this is the ability to reverse this process and find a curvature that generates the observed curved space values. With the advent of a universe simulator designed with the sole purpose of determining curvature parameters, future curvature measurements with new and old data alike will be faster and less computationally intense.

In its current state, CURVFAM has the ability to display the effects of various curvatures on any provided selection of data. This is useful in the testing of theories regarding the effects of curvature on systems. Later versions of CURVFAM will include a relativistic statistical fitting algorithm as well as a more user friendly GUI for streamlined data input and output. This will allow for the automated fitting of curvature parameters to large quantities of data. In the future, CURVFAM aims to generate a better model of the universe and increase our understanding of its topology both in space and in time.

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Software: Python
CURVFAM v1.0 available on GitHub: https://github.com/Breadcups/CURVFAM.git

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